Lecture 9: The Structure of the Proton: Part I

Sept 22, 2016

The Proton is Not a Point-like Particle

- Quark model says p consists of 3 quarks
  - Are they real?
- Gyromagnetic moment $g_p = 5.586$ is far from the Dirac value of 2 that holds for pointlike spin-$\frac{1}{2}$ particles
  - Pattern of baryon magnetic moments can be explained using quark model with fraction charges, fitting for quark masses
- Size of nucleus consistent with nucleons of size ~ 0.8 fm

To study structure of the proton, will use scattering techniques

Similar idea to Rutherford's initial discover of the nucleus

Scattering of Spinless Pointlike Particles

- Rutherford Scattering (spinless electron scattering from a static point charge) in lab frame:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4\left(\frac{1}{2}\theta\right)}$$

  here E is energy of incident electron and θ is scattering angle in the lab frame

- Mott Scattering: Taking into account statistics of identical spinless particles

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 \cos^2\left(\frac{1}{2}\theta\right)}{4E^2 \sin^4\left(\frac{1}{2}\theta\right)}$$

Scattering of Spin-$\frac{1}{2}$ Pointlike Particles

- Elastic Scattering of a spin-$\frac{1}{2}$ electron from a pointlike spin-$\frac{1}{2}$ particle of mass M:
  - Elastic scattering of electron from infinite mass target changes angle but not energy
  - For target of finite mass M, final electron energy is

$$E' = \frac{E}{1 + \frac{2E}{M}\sin^2\left(\frac{1}{2}\theta\right)}$$

and the four-momentum transfer is

$$q^2 = -4EE'\sin^2\left(\frac{1}{2}\theta\right)$$

The elastic scattering cross section is:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2\cos^2\left(\frac{1}{2}\theta\right)}{4E^2\sin^4\left(\frac{1}{2}\theta\right)}\frac{E'}{E}\left[1 - \frac{q^2}{2M^2}\tan^2\left(\frac{1}{2}\theta\right)\right]$$

What Happens if the Target Particles Have Finite Size?

- Charge distribution $\rho(r)$: $\int \rho(r) d^3r = 1$
- Scattering amplitude modified by a "Form Factor"

$$F(q^2) = \int d^3r e^{i\vec{q}\cdot\vec{r}} \rho(r)$$

  So that the cross section is modified by a factor of $|F(q^2)|^2$

- Note: As $q^2 \to 0$, $F(q^2) \to 1$
- We therefore can Taylor expand

$$F(q^2) = \int d^3r \left(1 + i\vec{q}\cdot\vec{r} - \frac{1}{2}(\vec{q}\cdot\vec{r})^2 + ...\right) \rho(r)$$

Form Factors

- The first $\vec{q} \cdot \vec{r}$ term vanishes when we integrate

$$
\begin{aligned}
F(q^2) &= 1 - \frac{1}{2} \int r^2 dr d\cos\theta d\phi \; \rho(r) (qr)^2 \cos^2\theta \\
&= 1 - \frac{2\pi}{2} \int dr d\cos\theta \; q^2 r^4 \cos^2\theta \\
&= 1 - \frac{< r^2 >}{4} q^2 \int \cos^2\theta \; d\cos\theta \\
&= 1 - \frac{< r^2 >}{4} q^2 \left[\frac{\cos^3\theta}{3} \right]_{-1}^{1} \\
&= 1 - \frac{< r^2 >}{6} q^2
\end{aligned}
$$

- For elastic scattering, can relate q to the outgoing angle

$$
q = \frac{2p \sin(\theta/2)}{\left[1 + (2E/M_p) \sin^2(\theta/2)\right]^{\frac{1}{2}}}
$$

where p and E are the momentum and energy of the incident electron in the lab frame

Hoffstader and McAllister (1956)

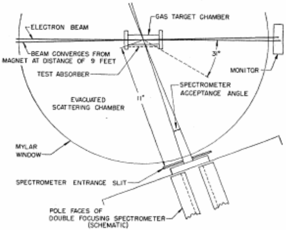


FIG. 2. Arrangement of parts in experiments on electron scattering from a gas target.

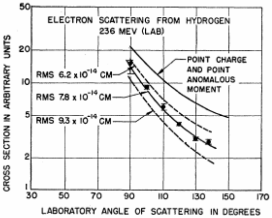


FIG. 6. This figure shows the experimental points at 236 Mev and the attempts to fit the shape of the experimental curve. The best fit lies near 0.78×10^{-13} cm.

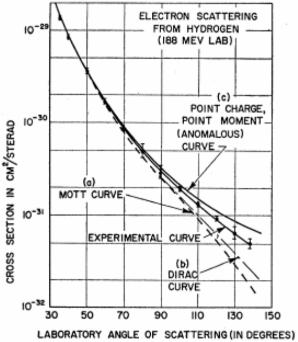


FIG. 5. Curve (a) shows the theoretical Mott curve for a spinless point proton. Curve (b) shows the theoretical curve for a point proton with the Dirac magnetic moment, curve (c) the theoretical curve for a point proton having the anomalous contribution in addition to the Dirac value of magnetic moment. The theoretical curves (b) and (c) are due to Rosenbluth.[8] The experimental curve falls between curves (b) and (c). This deviation from the theoretical curves represents the effect of a form factor for the proton and indicates structure within the proton, or alternatively, a breakdown of the Coulomb law. The best fit indicates a size of 0.70×10^{-13} cm.

$$< r^2 >^{\frac{1}{2}} = 0.74 \pm 0.24 \times 10^{-13} \text{ cm} \sim 0.7 \text{ fm}$$

Elastic Scattering: More on Angular Distributions (I)

- For elastic scattering, the angle uniquely determines the energy of the outgoing electron
  - So angle is the only independent variable
- Can write down the most general form of the matrix element

$$\mathcal{M} = \frac{2\pi\alpha}{q^2} J_\mu^{electron}(q) J^{\mu\ proton}(q)$$

  - Electron is a Dirac particle, so we know $J_\mu^{electron}(q) = e\overline{\psi}\gamma_\mu\psi$
  - For proton, write down the most general form allowed by Lorentz invariance and parity conservation

$$J_\mu^{proton} = \overline{\psi}(p_f)\left[F_1(q^2)\gamma_\mu + i\frac{q^\nu\sigma_{\mu\nu}\kappa}{2M}F_2(q^2)\right]\psi(p_i)$$

where p_i and p_f are the initial and final proton four-momenta, $q = k_i - k_f = p_f - p_i$ is the four-momentum transfer and κ is the anomalous magnetic moment of the proton

Elastic Scattering: Angular Distributions (II)

- Using the above, can calculate the cross section:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 \cos^2\left(\frac{1}{2}\theta\right)}{4E^2 \sin^4(\frac{1}{2}\theta)} \frac{E'}{E} \left[\left(F_1^2 + \frac{\kappa^2 Q^2}{4M^2} F_2^2 \right) + \frac{Q^2}{2M^2} \left(F_1 + \kappa F_2\right)^2 \tan^2\left(\frac{1}{2}\theta\right) \right]$$

- κ is the anomolous magnetic moment
- From slide 4, for pointlike spin-1/2 particles:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 \cos^2\left(\frac{1}{2}\theta\right)}{4E^2 \sin^4\left(\frac{1}{2}\theta\right)} \frac{E'}{E} \left[1 - \frac{q^2}{2M^2} \tan^2\left(\frac{1}{2}\theta\right) \right]$$

- Understanding these two form factors tells us about the structure of the proton

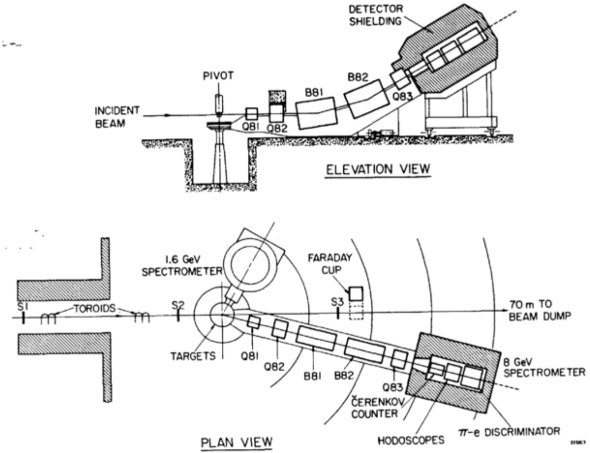
  We'll come back to this in a few minutes

Need High Energy Lepton Probe

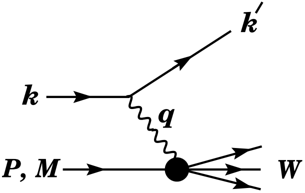
Stanford Linear Collider (SLAC)

- Two mile linear accelerator (e^-)
- Initial phase: energy = 20 GeV
- (Later, upgrade to 50 GeV)
- "End Station A" hall for fixed target experiments
- Study high momentum transfer
  - Need four-momentum transfer large enough to probe structure
  - Proton breaks apart
  - Deep Inelastic Scattering (DIS)

The SLAC-MIT DIS Experiment (1968)



Deep Inelastic Scattering



- W is the invariant mass of the hadronic system
- In lab frame: $P = (M, 0)$
- In any frame, $k = k' + q$, $W = p + q$
- Invariants of the problem:

$$
\begin{aligned}
Q^2 &= -q^2 = -(k-k')^2 \\
&= 2EE'(1-\cos\theta) \;\; [\text{in lab}] \\
P\cdot q &= P\cdot(k-k') \\
&= M(E-E') \;\; [\text{in lab}]
\end{aligned}
$$

- Define $\nu \equiv E - E'$ (in lab frame) so $P \cdot q = m\nu$ and

$$
\begin{aligned}
W^2 &= (P+q)^2 \\
&= (P-Q)^2 \\
&= M^2 + 2P\cdot q - Q^2 \\
&= M^2 + 2M\nu - Q^2
\end{aligned}
$$

where $Q^2 = -q^2$

- Elastic scattering corresponds to $W^2 = P^2 = M^2$
  - $Q^2 = 2M\nu$ elastic scattering
- We can define 2 indep dimensionless parameters

$$
\begin{aligned}
x &\equiv Q^2/2M\nu; \quad (0 < x \le 1) \\
y &\equiv \frac{P\cdot q}{P\cdot k} = 1 - E'/E; \quad (0 < y \le 1)
\end{aligned}
$$

The Most General Form of the Interaction

- Express cross section

$$d\sigma = L^e_{\mu\nu} W^{\mu\nu}$$

  where W describes the proton current (allowing substructure)

- Most general Lorentz invarient form of $W^{\mu\nu}$
  - Constructed from $g^{\mu\nu}$, p^μ and q^μ
  - Symmetric under interchange of μ and ν (otherwise vanishes when contracted with $L_{\mu\nu}$)

$$W^{\mu\nu} = -W_1 g^{\mu\nu} + \frac{W_2}{M^2} p^\mu p^\nu + \frac{W_4}{M^2} q^\mu q^\nu + \frac{W_5}{M^2}(p^\mu q^\nu + p^\nu q^\mu)$$

- W_3 reserved for parity violating term
- Not all 4 terms are independent. Using $\partial_\mu J^\mu = 0$ can show

$$\begin{aligned}
W_5 &= -\frac{p \cdot q}{q^2} W_2 \\
W_4 &= \frac{p \cdot q}{q^2} W_2 + \frac{M^2}{q^2} W_1 \\
W^{\mu\nu} &= W_1 (-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}) + W_2 \frac{1}{M^2} (p^\mu - \frac{p \cdot q}{q^2} q^\mu)(p^\nu - \frac{p \cdot q}{q^2} q^\nu)
\end{aligned}$$

Structure Functions

- Using notation from previous page, we can express the x-section for DIS

$$\frac{d\sigma}{d\Omega dE'} = \frac{\alpha^2}{4E^2}\frac{\cos^2(\frac{1}{2}\theta)}{\sin^4(\frac{1}{2}\theta)}\left[W_2(q^2,W) + 2W_1(q^2,W)\tan^2(\frac{1}{2}\theta)\right]$$

- These are the same two terms as for the elastic scattering
- W_1 and W_2 care called the *structure functions*
  - Angular dependence here comes from expressing covariant form on last page in lab frame variables
  - Two structure functions that each depend on Q^2 and W
  - Alternatively, can parameterize wrt dimensionless variables:

$$\begin{aligned} x &\equiv Q^2/2M\nu \\ y &\equiv \frac{P\cdot q}{P\cdot k} = 1 - E'/E \end{aligned}$$

Studying the Proton at Large Momentum Transfer

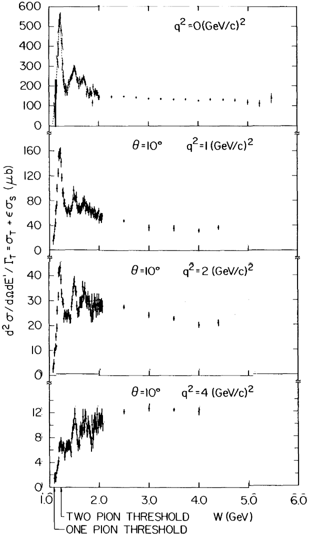
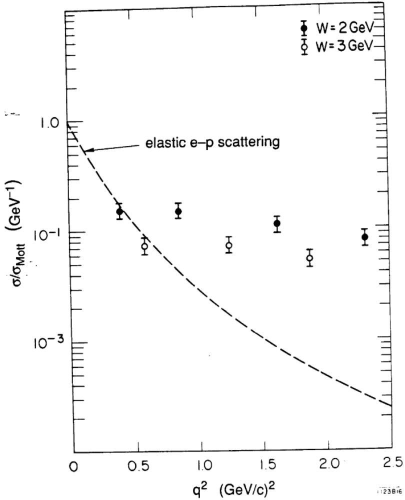


Fig. 9. Spectra of electrons scattered from hydrogen at q^2 up to 4 (GeV/c)^2. The curve for $q^2 = 0$ represents an extrapolation to $q^2 = 0$ of electron scattering data acquired at $\theta = 1.5°$. Elastic peaks have been subtracted and radiative corrections have been applied.

- SLAC-MIT group measured $d\sigma/dq^2 d\nu$ at 2 angles: $6°$ and $10°$
- For low W dominated by production of resonances
- Surprise: Above the resonance region, σ did *not* fall with Q^2
- Like Rutherford scattering, this is evidence for hard structure within the proton

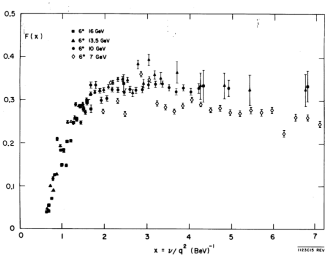
Evidence for Hard Substructure

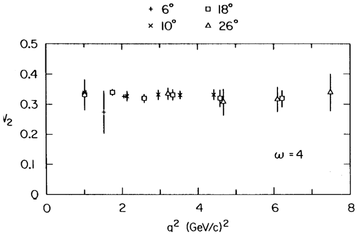
$$\frac{d\sigma}{d\Omega dE'} = \frac{\alpha^2}{4E^2}\frac{\cos^2(\frac{1}{2}\theta)}{\sin^4(\frac{1}{2}\theta)}\left[W_2(q^2,W) + 2W_1(q^2,W)\tan^2(\frac{1}{2}\theta)\right]$$



- How should we parameterize this deviation from behaviour predicted for pointlike proton?
  - To determine W_1 and W_2 separately, would need to measure at 2 values of E' and of θ that give the same q^2 and ν
  - The first exp couldn't do this: small angle where experiment ran, W_2 dominates so study that

SLAC-MIT Results: Scaling





- One more change of variables:

$$F_1(x, Q^2) \equiv MW_1(\nu, Q^2)$$
$$F_2(x, Q^2) \equiv \nu W_2(\nu, Q^2)$$

- Study F_2 for various energies and angles
- When low Q^2 data excluded, F_2 appears to depend only on dimensionless variable x and not on Q^2
- This phenomenon is called "scaling"

What does Scaling Tell Us? (I)

- Supposed there are pointlike partons inside the nucleon
- Work in an "infinite momentum" frame: ignore mass effects
- Proton 4-momentum: $P = (P, 0, 0, P)$
- Visualize stream of parallel partons each with 4-momentum xP where $0 < x < 1$; neglect transverse motion of the partons
  - x is the fraction of the proton's momentum that the parton carries
- If electron elastically scatters from a parton

$$\begin{aligned} (xP+q)^2 = m^2 &\simeq 0 \\ x^2P^2 + 2xP \cdot q + q^2 &= 0 \end{aligned}$$

Since $P^2 = M^2$, if $x^2M^2 << q^2$ then

$$\begin{aligned} 2xP \cdot q &= -q^2 = Q^2 \\ x &= \frac{Q^2}{2P \cdot q} = \frac{q^2}{2M\nu} \end{aligned}$$

This x is the same x we defined before!

Deep inelastic scattering can be described as elastic scattering of the lepton with a parton with momentum xP

What does Scaling Tell Us? (II)

- Suppose the partons in the proton have a distribution of fractional momentum $f(x)$
  - $f(x)$dx $=$ probability of finding a parton carrying a fraction of the proton's momentum between x and $x = dx$
- We can write the inelastic ep scattering cross section as an incoherent sum of elastic scatters off the partons inside the proton

$$\frac{d\sigma^{ep}}{dEd\Omega} = \sum_i \int_0^1 f(x) \frac{d\sigma^{ei}}{dEd\Omega}$$

  where the sum over i is a sum over partons
- The cross section only depends on $x = q^2/2M\nu$ because that is the combination that picks out the momentum fraction carried by the parton

What Have We Learned?

Scaling of the Structure Functions is evidence for the presence of pointlike partons with the proton!

Some comments:

- We are using an impulse approximation where the scattering occurs before the partons have a chance to redistribute themselves
- We implicitly assume that after the scattering, the partons that participate in the scattering turn into hadrons with probability=1
- This is a lowest order calculation. We will see later that to higher order in perturbation theory, QCD corrections will introduce slow scaling violations (Q^2 dependence).

Some Observations (I)

- Let $f(x)$ be the prob of finding a parton with mom fraction between x and $x + dx$ in the proton.
- If the partons together carry all the momentum of the proton

$$\int dx\, x f(x) = \int dx\, x \sum_i f_i(x) = 1$$

  where $\sum_i$ is a sum over *all* partons in the proton
- We call $f(x)$ the parton distribution function since it tells us the momentum distribution of the parton within the proton
- This is the first example of a "sum rule"

Some Observations (II)

- It's natural to associate the partons with quarks, but that's not the whole story
- Because ep scattering occurs through the electromagnetic interaction, it only occurs via scattering with charged partons.
- If the proton also contains neutral partons, the EM scattering won't "see" them
- Let's assume that the ep scattering occurs through the scattering of the e off a quark or antiquark
  - We saw that the SU(3) description of the proton consists of $2\ u$ and $1\ d$ quark.
  - However we can in addition have any number of $q\bar{q}$ pairs without changing the proton's quantum numbers
  - The 3 quarks (uud) are called *valence quarks*. The additional $q\bar{q}$ pairs are called *sea* or *ocean* quarks.

Another Sum Rule

- To get the right quark content for the proton:

$$\begin{aligned} \int u(x) - \overline{u}(x) dx &= 2 \\ \int d(x) - \overline{d}(x) dx &= 1 \\ \int s(x) - \overline{s}(x) dx &= 0 \end{aligned}$$

Writing the DIS cross section in terms of PDFs

- The cross section is incoherent sum over elastic scattering with partons
- If partons are quarks, they are Dirac particles and we can calculate everything:

$$\frac{d\sigma}{dE'd\Omega}|_{Dirac} = \frac{4\alpha^2 E'^2}{Q^2}\left[\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right]\delta(\nu + q^2/2Mx)$$

- Taking incoherent sums:

$$\frac{d\sigma}{dE'd\Omega}|_{Dirac} = \sum_i \int_0^1 dx\ f_i(x)e_i^2\left[\cos^2\frac{\theta}{2} + Q^2/2M^2x^2\sin^2\frac{\theta}{2}\right]\delta(\nu + q^2/2Mx)$$

- Using the property of delta functions

$$\delta(g(x)) = \frac{\delta(x - x')}{|g'(x - x_0)|_{x=x_0}}$$

where $g(x_0) = 0$, we can write

$$\delta(\nu - Q^2/2Mx) = \frac{\delta(x - Q^2/2M\nu)}{Q^2/2Mx^2} = \frac{x}{v}\delta(x - Q^2/2M\nu)$$

- This gives us:

$$\frac{d\sigma}{dE'd\Omega} = \sum_i \int_0^1 dx\ f_i(x)e_i^2\left[\cos^2\frac{\theta}{2} + \frac{Q^2}{2M^2x^2}\sin^2\frac{\theta}{2}\right]\frac{x}{\nu}\ \delta(x - Q^2/2M\nu)$$

Continuing from last page

- The result we just obtained on the previous page was

$$\frac{d\sigma}{dE' d\Omega} = \sum_i \int_0^1 dx\; f_i(x) e_i^2 \left[\cos^2\frac{\theta}{2} + \frac{Q^2}{2M^2x^2}\sin^2\frac{\theta}{2}\right] \frac{x}{\nu}\delta(x - Q^2/2M\nu)$$

- Notice that we have one term proportional to $\cos^2\frac{\theta}{2}$ and one proportional to $\sin^2\frac{\theta}{2}$
  - This is the same form as our phenomenological form for the ep scattering:

$$\frac{d\sigma}{dE' d\Omega} = \frac{\alpha^2}{4E^2}\frac{1}{\sin^4(\frac{\theta}{2})}\left[W_2(x)\cos^2(\frac{\theta}{2}) + 2W_1(x)\sin^2(\frac{\theta}{2})\right]$$

  - Equating terms

$$\begin{aligned} W_2(x) &= \sum_i f_i(x) e_i^2 \frac{x}{\nu} \\ W_1(x) &= \sum_i f_i(x) e_i^2 \frac{Q^2}{2M\nu}\frac{x}{\nu}\delta(x - \frac{Q^2}{2M\nu}) \\ &= \sum_i f_i(x)\frac{e_i^2}{2M} \end{aligned}$$

- It's usual to define

$$F_2(x) \equiv \nu W_2(x) \qquad F_1(x) \equiv M W_1(x)$$

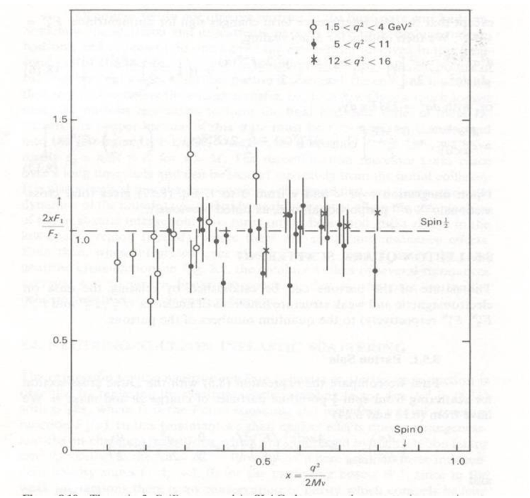
  Then

$$F_2(x) = \sum_i x\; f_i(x) e_i^2 \qquad F_1(x) = \textstyle\sum_i f_i(x)\frac{e_i^2}{2}$$

$$F_2(x) = 2xF_1(x)$$

  This is called the Callen-Gross relation
- Note: If our partons had spin-0 rather than spin-$\frac{1}{2}$, we would have found $F_1 = 0$

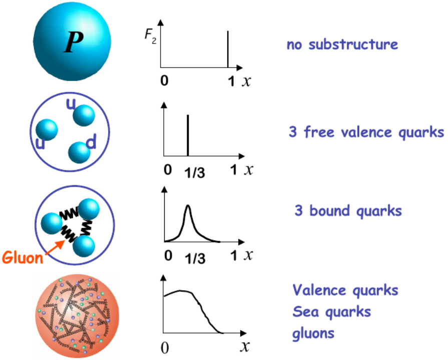
What does the data look like?



from Perkins

The partons act like spin-1/2 Dirac particles!

If partons are quarks, what do we expect?



Max Klein, CTEQ School Rhodos 2006

- Elastic scattering from proton has $x = 1$
- If 3 quarks carry all the proton's momentum each has $x = 0.3$
- Interactions among quarks smears $f(x)$
- Radiation of gluons softens distribution and adds $q\bar{q}$ pairs
  - Describe the 3 original quarks as "valence quarks"
  - $q\bar{q}$ pairs as sea or ocean
- Some of proton's momentum carried by gluons and not quarks or antiquarks

Using Isospin: Comparing the Proton and Neutron

- Ignore heavy quark content in the proton: consider only u, d, s
- Write the proton Structure Function

$$\frac{F_2^p(x)}{x} = \sum_i f_i^p(x) e_i^2 = \frac{4}{9}(u^p(x) + \overline{u}^p(x)) + \frac{1}{9}(d^p(x) + \overline{d}^p(x)) + \frac{1}{9}(s^p(x) + \overline{s}^p(x))$$

- Similarly, for the neutron

$$\frac{F_2^n(x)}{x} = \sum_i f_i^n(x) e_i^2 = \frac{4}{9}(u^n(x) + \overline{u}^n(x)) + \frac{1}{9}(d^n(x) + \overline{d}^n(x)) + \frac{1}{9}(s^n(x) + \overline{s}^n(x))$$

- But isospin invariance tells us that $u^p(x) = d^n(x)$ and $d^p(x) = u^n(x)$
- Write F_2 for the neutron in terms of the proton pdf's (assuming same strange content for the proton and neutron)

$$\frac{F_2^n(x)}{x} = \frac{4}{9}(d^p(x) + \overline{d}^p(x)) + \frac{1}{9}(u^p(x) + \overline{u}^p(x)) + \frac{1}{9}(s^p(x) + \overline{s}^p(x))$$

- Assuming sea q and $\overline{q}$ distributions are the same:

$$u(x) - \overline{u}(x) = u_v(x), \quad d(x) - \overline{d}(x) = d_v(x), \quad s(x) - \overline{s}(x) = 0$$

- Taking the difference in F_2 for protons and neutrons:

$$\frac{1}{x}[F_2^p(x) - F_2^n(x)] = \frac{1}{3}[u_v(x) - d_v(x)]$$

which gives us a feel for the valence quark distribution

What the data tells us

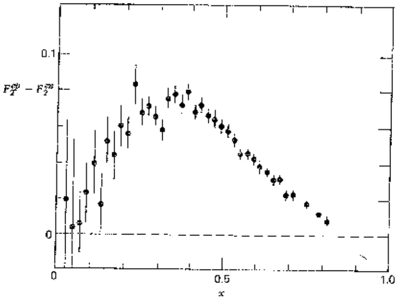


Fig. 9.8 The difference $F_2^{ep} - F_2^{en}$ as a function of x, as measured in deep inelastic scattering. Data are from the Stanford Linear Accelerator.

From Halzen and Martin

- Looks the way we expect from the cartoon on page 27
- Next question: How to measure the partons' charge
  - To do this, must compare e and ν scattering!